Couplings of gravitational currents with Chern-Simons gravities

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Coupling of conserved p-brane currents with non-abelian gauge theories is done consistently by using Chern-Simons forms. Conserved currents localized on p-branes that have gravitational origin can be constructed from Killing-Yano forms of underlying spacetime. We propose a generalization of the coupling procedure with Chern-Simons gravities to the gravitational conserved currents case. In odd dimensions, the field equations of coupled Chern-Simons gravities which describe the local curvature on p-branes are obtained. In special cases of three and five dimensions, the field equations are investigated in detail.

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I. INTRODUCTION

Couplings with external sources in gauge theories are described by the well-known minimal coupling procedure. However, this is relevant only for the external point sources and in the case of extended objects the situation is different. Extended objects that have p space dimensions are called p-branes and they have $(p+1)$-dimensional worldvolumes. p-branes are the generalizations of the point particles to higher dimensions. Charges that are localized on p-branes define conserved currents in spacetime. Coupling of these currents with non-abelian gauge theories in the standard minimal coupling procedure is problematic [1, 2]. However, in the case of Chern-Simons (CS) gauge theories, problem of coupling with extended sources have a natural solution.

CS theories of non-abelian gauge fields are metric-free and background independent gauge theories that exist in odd dimensions. CS theories of gravity are also defined in odd dimensions by using the de Sitter (or anti-de Sitter) gauge connections in the first order formalism of gravity [3]. Coupling of extended sources with CS gauge theories generalize the minimal coupling procedure by using the transformation properties of CS forms under gauge transformations. CS forms transform as abelian gauge connections and this property produces a consistent gauge invariant coupling between a conserved current and a CS form. The interaction term in the action is gauge invariant up to a boundary term if the coupling current is conserved. So the conserved currents on $2p$-branes can couple with CS gravities consistently. However, CS theories can only be defined in odd dimensions and hence only the coupling of even dimensional branes can be written. Even dimensional branes and odd dimensional CS forms define a consistent interaction term in the action. One example for conserved currents is the electromagnetic currents that are defined on p-brane worldvolumes. Coupling between electromagnetic extended sources and CS gravities are recently studied in the literature [4, 5]. Coupling of extended charged events with CS theories is also considered in [6].

Conserved charges in theories of gravitation can be defined from the asymptotical symmetries of the spacetime. Killing vector fields of asymptotically flat or asymptotically AdS spacetimes are used in defining the mass and angular momentum in general relativity [7]. On the other hand, for extended objects like p-branes, the definition of conserved currents can be generalized by using Killing-Yano (KY) forms [8, 9]. KY forms define hidden symmetries of spacetime that are generalizations of Killing vector fields to higher order forms [10]. Conserved currents that are constructed from KY forms are localized on p-branes and conserved charges for these branes can be defined by using the asymptotical symmetries of transverse directions to the brane. Conservation of the currents constructed from KY forms are shown in [9].

Conserved gravitational currents can also consistently couple with CS gravities. Currents localized on p-branes affect the local geometry of the brane and the field equations of CS gravities coupled with gravitational currents give this local geometry. CS gravities which have globally AdS structure induces conserved currents on p-branes from KY forms of AdS spacetime. Because CS gravities are defined in odd dimensions, the KY forms that have odd form degrees are used in the construction of these currents. In this work, we generalize the coupling of conserved currents with CS gravities to the gravitational currents case. We find the field equations that define the local geometry of $2p$-branes and give special examples in three and five dimensions.

The paper is organized as follows. In Section 2, we review the electromagnetic current couplings of $2p$-branes with CS gravities. Definition of KY forms and construction of conserved gravitational currents from KY forms are included in Section 3. Couplings of gravitational currents with CS gravities in arbitrary odd dimensions are considered in Section 4. Section 5 presents special exam-

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ples for three and five dimensions and the conclusion is given in Section 6.

II. BRANE COUPLINGS IN CS THEORIES

Coupling of conserved currents with gauge connections in n dimensions is provided by the minimal coupling term in the action

$$I_{MC} = \int_{M^n} d^n x j_\mu A^\mu \tag{1}$$

where j_μ is the conserved current generated by a point source and A^μ is the vector potential. Generalization of minimal coupling procedure to extended sources like p-branes is possible for abelian connections in the form $j_{\mu_1\mu_2...\mu_p}A^{\mu_1\mu_2...\mu_p}$. However, for non-abelian connections this procedure is not well-defined [1, 2]. In the general case, the couplings of extended objects are described gauge invariantly by using CS forms. A p-brane is defined as an object that extends to p space dimensions and it has a $(p+1)$ dimensional worldvolume. The currents localized on p-brane worldvolumes are defined by the transverse directions to the brane. Hence, in $2n+1$ dimensions, the current localized on a $2p$-brane is a $2n+1-(2p+1) = 2(n-p)$-form.

Let $\mathbf{A}$ be a non-abelian gauge connection which is a Lie algebra valued 1-form. The connection transforms under gauge transformations as follows;

$$\mathbf{A} \to \mathbf{A}' = g^{-1}\mathbf{A}g + g^{-1}dg \tag{2}$$

where g is an arbitrary element of the Lie group. In $2n+1$ dimensions, CS forms are defined from the connection $\mathbf{A}$ as [11];

$$\begin{aligned}\langle \mathcal{C}_{2n+1}(\mathbf{A})\rangle &= \frac{1}{n+1}\langle \mathbf{A}(d\mathbf{A})^n \\ &\quad + c_1\mathbf{A}^3(d\mathbf{A})^{n-1} + ... + c_n\mathbf{A}^{2n+1}\rangle\end{aligned} \tag{3}$$

where $\langle \quad \rangle$ denotes the invariant symmetric trace namely the Cartan-Killing form in the Lie algebra which takes traces of the Lie algebra elements in the adjoint representation and $\mathbf{A}^n = \mathbf{A}\wedge ... \wedge\mathbf{A}$ (n times). $c_1, ..., c_n$ are dimensionless coefficients determined by the condition;

$$d\langle \mathcal{C}_{2n+1}(\mathbf{A})\rangle = \frac{1}{n+1}\langle \mathbf{F}^{n+1}\rangle. \tag{4}$$

Here d is the exterior derivative operator and $\mathbf{F} = d\mathbf{A} + \mathbf{A}\wedge\mathbf{A}$ is the curvature of the connection $\mathbf{A}$. CS forms transform under gauge transformations as abelian connections [3];

$$\mathcal{C}_{2p+1}(\mathbf{A}') \to \mathcal{C}_{2p+1}(\mathbf{A}) + d\Omega_{(2p)} \tag{5}$$

where $p = 0, ..., n$ and $\Omega_{(2p)}$ is an arbitrary $2p$-form. This property is responsible for the consistent coupling between conserved currents and CS forms; because the coupling term in the action is

$$I_C = \int \langle j_{(2n-2p)} \wedge \mathcal{C}_{2p+1}(\mathbf{A})\rangle \tag{6}$$

and remains gauge invariant up to a boundary term. Here $j_{(2n-2p)}$ is a conserved current localized on a $2p$-brane.

The action of CS gauge theories in $2n+1$ dimensions is defined as follows;

$$I_{CS} = \kappa \int_{M^{2n+1}} \langle \mathcal{C}_{2n+1}(\mathbf{A})\rangle \tag{7}$$

where κ is a dimensionless constant. A CS theory can couple with a conserved $(2n-2p)$-form current localized on a $2p$-brane. The total action for CS theory coupled with a $2p$-brane is

$$I_{2n+1} = \kappa \int_{M^{2n+1}} \langle \mathcal{C}_{2n+1}(\mathbf{A}) - j_{(2n-2p)} \wedge \mathcal{C}_{2p+1}(\mathbf{A})\rangle. \tag{8}$$

The field equations

$$\mathbf{F}^n = j_{(2n-2p)} \wedge \mathbf{F}^p. \tag{9}$$

can be found by varying the action with respect to $\mathbf{A}$. Thus, outside the worldvolume of the brane the field equations are $\mathbf{F}^n = 0$. However, on the worldvolume, different solutions appear. For example, an electromagnetic current on a $2p$-brane can be defined as

$$j_{(2n-2p)} = q_{2p}\delta(T^{2n-2p})d\Omega^{2n-2p}\mathbf{G}^{\mathbf{J}_1...\mathbf{J}_{n-p}} \tag{10}$$

where q_{2p} is the electric charge on the brane, $\delta(T^{2n-2p})$ denotes the localization of the current on the transverse directions T^{2n-2p} to the brane and $d\Omega^{2n-2p}$ is the volume form on the transverse directions to the brane. $\mathbf{G}^{\mathbf{J}_1...\mathbf{J}_{n-p}}$ is constructed from the Lie algebra generators $\mathbf{J}_1, ..., \mathbf{J}_{n-p}$. Hence, the current is written as a Lie algebra valued $2(n-p)$-form [4]. This conserved current defines a non-trivial curvature on the brane through the field equations.

III. KY FORMS AND GRAVITATIONAL CURRENTS

Conserved quantities in gravitational theories are described by the symmetries of the underlying spacetime. If spacetime has Killing vector fields, which generate local isometries of the spacetime, then a conserved current can be constructed by using them. The well-known gravitational 1-form current is written as $j_{(1)} = K_a *^{-1} G^a$ where K_a are the components of a Killing vector field K, $*^{-1}$ is the inverse Hodge map on differential forms and G^a are the Einstein $(n-1)$-forms in n dimensions. Corresponding conserved charges are defined from the asymptotical symmetries of the spacetime. Generalization of gravitational conserved currents can be obtained by using KY

forms which generalize the Killing vector fields to higher order forms.

If $\omega_{(p)}$ is a KY p-form then it satisfies the following equation

$$\nabla_X \omega_{(p)} = \frac{1}{p+1} i_X d\omega_{(p)} \tag{11}$$

for all vector fields X, which is the generalization of Killing's equation. Here ∇_X is the covariant derivative and i_X is the interior derivative (contraction) operator with respect to the vector field X. This equation implies that all KY forms are co-closed, namely $\delta\omega_{(p)} = 0$ where $\delta = (-1)^p *^{-1} d*$ is the co-derivative operator and $*$ is the Hodge map on differential forms. For a class of spherically symmetric spacetimes, solutions of the KY equation in four dimensions are found in [10].

Two basic conserved gravitational currents can be defined from the curvature characteristics and KY forms $\omega_{(p)}$ of the underlying spacetime. The first current is defined as;

$$\begin{aligned} \mathcal{J}_1(\omega_{(p)}) &= i_{X_a}(i_{X_b}\omega_{(p)} \wedge R^{ba}) \\ &= -i_{X_a} i_{X_b}\omega_{(p)} \wedge R^{ab} + (-1)^p i_{X_a}\omega_{(p)} \wedge P^a \end{aligned} \tag{12}$$

and the second one is

$$\begin{aligned} \mathcal{J}_2(\omega_{(p)}) &= (-1)^p i_{X_a}(\omega_{(p)} \wedge P^a) \\ &= (-1)^p i_{X_a}\omega_{(p)} \wedge P^a + \mathcal{R}\omega_{(p)} \end{aligned} \tag{13}$$

where R^{ab} are curvature 2-forms, $P^a = i_{X_b} R^{ba}$ are Ricci 1-forms and $\mathcal{R} = i_{X_a} P^a$ is the curvature scalar and X_a is an arbitrary frame basis. We will use $\mathcal{J}_1$ and $\mathcal{J}_2$ instead of $\mathcal{J}_1(\omega_{(p)})$ and $\mathcal{J}_2(\omega_{(p)})$ for brevity. As it is shown in [9], both of these currents are co-closed;

$$\delta\mathcal{J}_1 = 0 = \delta\mathcal{J}_2 \tag{14}$$

hence the currents $*\mathcal{J}_1$ and $*\mathcal{J}_2$ are conserved.

The term 'gravitational currents' indeed means that they are defined from curvature characteristics and hidden symmetries of the background spacetime and there is no direct relation between the currents and Einstein field equations. Hence, they can be seen as analogous to the electromagnetic currents in some sense, though they are different by their way of construction. So, these currents can be interpreted as they are localized on p-branes and can define charge densities for p-brane spacetimes. This opens the possibility of coupling of gravitational conserved currents on p-branes with CS gravities.

As a special case, the currents have more simple forms in constant curvature spacetimes. The curvature characteristics of an n-dimensional constant curvature spacetime are given by the following equalities;

$$\begin{aligned} R^{ab} &= c e^a \wedge e^b \\ P^a &= c(n-1)e^a \\ \mathcal{R} &= cn(n-1) \end{aligned} \tag{15}$$

where c is a constant. Hence the currents defined in (12) and (13) can be written as constant multiples of KY p-forms;

$$\mathcal{J}_1 = -cp(n-p)\omega_{(p)} \tag{16}$$

$$\mathcal{J}_2 = c(n-1)(n-p)\omega_{(p)}. \tag{17}$$

Thus, they are linearly dependent and the conservation of their Hodge duals is a result of the co-closedness of KY forms. In an n-dimensional spacetime, the maximal number of KY p-forms is given by the number

$$C\,(n+1, p+1) = \frac{(n+1)!}{(p+1)!(n-p)!} \tag{18}$$

and this number is attained in constant curvature spacetimes. Hence, the number of KY p-forms in constant curvature spacetimes gives the number of independent gravitational conserved currents constructed from KY p-forms.

IV. COUPLINGS OF KY CURRENTS WITH CS GRAVITIES

In the first-order formalism of gravity, the fundamental fields that describe gravitational interactions are the co-frame 1-forms e^a and the connection 1-forms ω^{ab}. In $n+1$ dimensions these two quantities can be combined into a single Lie algebra valued gauge connection to construct the AdS ($SO(n-1,2)$) (or dS ($SO(n,1)$)) gauge theories of gravity [12];

$$\mathbf{A} = \frac{1}{2}\omega^{ab}\mathbf{J}_{ab} + \frac{1}{l} e^a \mathbf{J}_a \tag{19}$$

where $a, b = 0, 1, ..., n$ and l is a constant in units of length. $\mathbf{J}_{ab}$ and $\mathbf{J}_a = \mathbf{J}_{an}$ are the generators of the AdS algebra. The associated gauge curvature 2-form is written in terms of Riemann curvature 2-forms $R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$ and torsion 2-forms $T^a = de^a + \omega^a{}_b \wedge e^b$ as;

$$\begin{aligned} \mathbf{F} &= d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} \\ &= \frac{1}{2}\left(R^{ab} + \frac{1}{l^2} e^a \wedge e^b\right)\mathbf{J}_{ab} + \frac{1}{l} T^a \mathbf{J}_a. \end{aligned} \tag{20}$$

The flat connection $\mathbf{F} = 0$ corresponds to torsion-free, constant curvature AdS spacetime: $R^{ab} = -\frac{1}{l^2} e^a \wedge e^b$. From now on we take torsion to be zero.

CS gravities are defined from the AdS connection in $2n+1$ dimensions and the action that includes a coupling term with a current localized on a $2p$-brane is written as in (8);

$$I_{2n+1} = \kappa \int_{M^{2n+1}} \langle \mathcal{C}_{2n+1}(\mathbf{A}) - j_{(2n-2p)} \wedge \mathcal{C}_{2p+1}(\mathbf{A}) \rangle.$$

From the field equations (9) of this action;

$$\mathbf{F}^n = j_{(2n-2p)} \wedge \mathbf{F}^p,$$

it can be seen that in spacetime regions out of the brane, the field equations has the form $\mathbf{F}^n = 0$ and the solutions give the global structure of spacetime (one solution is $\mathbf{F} = 0$ and this implies the global AdS structure and this equation can also be satisfied by decomposable $\mathbf{F}$'s). This global structure defines the conserved gravitational currents localized on $2p$-branes. That property resembles the Mach's principle in gravitation theories [13] which states that the local motion of a body is determined by the large scale distribution of matter. So, the currents defined in (12) and (13) are constructed from curvature characteristics and KY forms of source-free regions of spacetime.

For the current $\mathcal{J}_1$ defined in (12), the action becomes

$$\begin{aligned} I_{2n+1} &= \kappa \int_{M^{2n+1}} \langle \mathcal{C}_{2n+1}(\mathbf{A}) - *\mathcal{J}_1 \wedge \mathcal{C}_{2p+1}(\mathbf{A}) \rangle \\ &= \kappa \int_{M^{2n+1}} \langle \mathcal{C}_{2n+1}(\mathbf{A}) \\ &- *(i_{X_a}(i_{X_b}\omega_{(2p+1)} \wedge R_G^{ba})) \wedge \mathcal{C}_{2p+1}(\mathbf{A}) \rangle \end{aligned} \quad (21)$$

and the field equations are

$$\mathbf{F}^n = *(i_{X_a}(i_{X_b}\omega_{(2p+1)} \wedge R_G^{ba})) \wedge \mathbf{F}^p \quad (22)$$

where the KY forms $\omega_{(2p+1)}$ and curvature 2-forms R_G^{ab} in the current are the characteristics of the global spacetime. From the definition of gauge curvature 2-form in (20), the wedge product of two curvature forms can be found as

$$\begin{aligned} \mathbf{F} \wedge \mathbf{F} &= \frac{1}{4}\left(R^{ab} + \frac{1}{l^2} e^a \wedge e^b\right) \\ &\wedge \left(R^{cd} + \frac{1}{l^2} e^c \wedge e^d\right) [\mathbf{J}_{ab}, \mathbf{J}_{cd}] \end{aligned}$$

and the field equations are written as follows;

$$\begin{aligned} &\frac{1}{2^n}\left(R^{ab} + \frac{1}{l^2} e^a \wedge e^b\right) \\ &\underbrace{\wedge ... \wedge}_{n-1} \left(R^{kl} + \frac{1}{l^2} e^k \wedge e^l\right) [\mathbf{J}_{ab}, ..., \mathbf{J}_{kl}] \\ &= *(i_{X_a}(i_{X_b}\omega_{(2p+1)} \wedge R_G^{ba}))\mathbf{J}_{12}...\mathbf{J}_{(n-p-1)(n-p)} \\ &\wedge \frac{1}{2^p}\left(R^{ab} + \frac{1}{l^2} e^a \wedge e^b\right) \\ &\underbrace{\wedge ... \wedge}_{p-1} \left(R^{pq} + \frac{1}{l^2} e^p \wedge e^q\right) [\mathbf{J}_{ab}, ..., \mathbf{J}_{pq}] \end{aligned}$$

where $[\mathbf{J}_{ab}, ..., \mathbf{J}_{kl}]$ denotes the commutator of Lie algebra generators which comes from the wedge product of Lie algebra valued forms.

For the current $\mathcal{J}_2$ defined in (13), the field equations become

$$\mathbf{F}^n = *(-i_{X_a}(\omega_{(2p+1)} \wedge P_G^a)) \wedge \mathbf{F}^p \quad (23)$$

and the same procedure applies for the multiple wedge products of gauge curvature 2-forms as in the first case.

Linear combinations of two currents are also conserved and they can couple with CS gravity. The field equations are found from the following action

$$\begin{aligned} I_{2n+1} &= \kappa \int_{M^{2n+1}} \langle \mathcal{C}_{2n+1}(\mathbf{A}) \\ &- \sum_{p=0}^{n-1} *(a_1\mathcal{J}_1 + a_2\mathcal{J}_2) \wedge \mathcal{C}_{2p+1}(\mathbf{A}) \rangle \end{aligned}$$

where a_1 and a_2 are arbitrary constants.

As a special case, in $2n + 1$ dimensions a conserved current localized on a $2(n-1)$-brane which is a 2-form leads to the field equations;

$$\mathbf{F}^{n-1} \wedge (\mathbf{F} - j) = 0. \quad (24)$$

This implies that two special solutions for this case are $\mathbf{F} = 0$ and $\mathbf{F} = j$. Hence, 2-form currents may not change the AdS curvature on the brane or current itself define the localized curvature on the brane.

V. SPECIAL CASES

Now, we consider the couplings of gravitational currents with CS gravities in three and five dimensions and find the exact field equations for them. These will give the local curvatures on the branes that are induced by gravitational currents constructed from the hidden symmetries of the global spacetime.

A. Brane Couplings in 3D

In three dimensions CS gravity action is equivalent to the three dimensional Einstein gravity with cosmological constant. CS action with a coupling term is written in this case as;

$$I_3 = \int_{M^3} \langle \mathcal{C}_3(\mathbf{A}) - j_{(2)} \wedge \mathcal{C}_1(\mathbf{A}) \rangle \quad (25)$$

where $\mathcal{C}_3(\mathbf{A}) = \frac{1}{2}(\mathbf{A} \wedge d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A})$ and $\mathcal{C}_1(\mathbf{A}) = \mathbf{A}$. Hence the action is

$$I_3 = \int_{M^3} \langle \frac{1}{2}(\mathbf{A} \wedge d\mathbf{A} + \mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}) - j_{(2)} \wedge \mathbf{A} \rangle \quad (26)$$

and the corresponding field equations are

$$\mathbf{F} = j_{(2)}. \quad (27)$$

In source free regions, this equation reduces to $\mathbf{F} = 0$ and this implies that the spacetime has global AdS structure $R^{ab} = -\frac{1}{l^2} e^a \wedge e^b$.

For the first KY current $\mathcal{J}_1$, by using (20) the field equations (27) transform into

$$\left(R^{ab} + \frac{1}{l^2} e^a \wedge e^b\right) \mathbf{J}_{ab} = -2 * \left(i_{X_c}\omega_{(1)} \wedge P^c_{AdS}\right)^{ab} \mathbf{J}_{ab} \quad (28)$$

In three dimensions, curvature 2-forms can be written in terms of Ricci 1-forms and curvature scalar [14];

$$R^{ab} = \frac{1}{2}\mathcal{R} e^b \wedge e^a + P^a \wedge e^b - P^b \wedge e^a. \quad (29)$$

Hence the field equations are written in the form

$$\begin{aligned} &\left(P^a \wedge e^b - P^b \wedge e^a - \frac{1}{2}\left(\mathcal{R} - \frac{2}{l^2}\right) e^a \wedge e^b\right) \mathbf{J}_{ab} \\ =\ &\left[\left((i_{X_c}\omega_{(1)}) i_{X_l} i_{X_k} * P^c_{AdS}\right) e^k_{AdS} \wedge e^l_{AdS}\right]^{ab} \mathbf{J}_{ab} \end{aligned}$$

where the equality in n dimensions $(-1)^{n-1}(*\phi) \wedge \widetilde{X} = *i_X \phi$ is used for an arbitrary form ϕ and $\widetilde{X}$ is the 1-form which is the metric dual of the vector field X. This can be written more compactly as

$$\begin{aligned} &\left(P^{[a} \wedge e^{b]} - \frac{1}{2}\left(\mathcal{R} - \frac{2}{l^2}\right) e^a \wedge e^b\right) \mathbf{J}_{ab} \\ =\ &\left[\left(\epsilon_{ckl}(i_{\widetilde{\omega_{(1)}}} P^c_{AdS})\right) e^k_{AdS} \wedge e^l_{AdS}\right]^{ab} \mathbf{J}_{ab} \end{aligned}$$

where [ ] on the indices denotes antisymmetrization and ϵ_{ckl} is the completely antisymmetric Levi-Civita symbol.

Curvature 2-forms of global AdS spacetime are $R^{ab}_{AdS} = -\frac{1}{l^2} e^a \wedge e^b$ and the Ricci 1-forms and the curvature scalar are obtained as

$$\begin{aligned} P^a_{AdS} &= -\frac{2}{l^2} e^a \\ \mathcal{R}_{AdS} &= -\frac{6}{l^2} \end{aligned}$$

and from the relation (16) the Hodge dual of the KY current $\mathcal{J}_1$ reduces to

$$* \mathcal{J}_1 = \frac{2}{l^2} * \omega_{(1)} \quad (30)$$

in AdS spacetime.

Let us write the KY 1-form $\omega_{(1)}$ in the coframe basis as follows;

$$\omega_{(1)} = \alpha e^0_{AdS} + \beta e^1_{AdS} + \gamma e^2_{AdS} \quad (31)$$

where α, β and γ are functions which are determined from the KY equation (11) for the AdS background. By using this definition and the curvature characteristics of AdS in (28), the field equations in three dimensions are found as below:

$$\begin{aligned} R^{01} + \frac{1}{l^2} e^0 \wedge e^1 &= -\frac{2\gamma}{l^2} e^0_{AdS} \wedge e^1_{AdS}, \\ R^{02} + \frac{1}{l^2} e^0 \wedge e^2 &= \frac{2\beta}{l^2} e^0_{AdS} \wedge e^2_{AdS}, \\ R^{12} + \frac{1}{l^2} e^1 \wedge e^2 &= \frac{2\alpha}{l^2} e^1_{AdS} \wedge e^2_{AdS}. \end{aligned} \quad (32)$$

Hence the local curvature around the brane is determined from the KY 1-forms of the global AdS spacetime. Curvature 2-forms of the brane that differ from AdS are given as corrections to the AdS background by KY form components.

KY forms of three dimensional AdS spacetime are given in Appendix A. Let us take the KY 1-form ω_3 in (A7) as an example. Then the field equations are written as

$$\begin{aligned} &R^{01} + \frac{1}{l^2} e^0 \wedge e^1 \\ &= \frac{2\kappa}{l^2}\left(\frac{r^2}{l^2} + 1\right)^{1/2} \cosh(\kappa t) \sin\phi dt \wedge dr, \\ &R^{02} + \frac{1}{l^2} e^0 \wedge e^2 \\ &= \frac{2\kappa r}{l^2}\left(\frac{r^2}{l^2} + 1\right)^{1/2} \sinh(\kappa t) \cos\phi dt \wedge d\phi, \\ &R^{12} + \frac{1}{l^2} e^1 \wedge e^2 \\ &= \frac{2r^2}{l^4}\left(\frac{r^2}{l^2} + 1\right)^{-1/2} \cosh(\kappa t) \cos\phi dr \wedge d\phi \end{aligned} \quad (33)$$

and solutions of these equations give the local co-frame on the worldvolume of the brane. In three dimensions, only 0-branes can couple consistently with CS theories as can be seen from the action (25). The wordline of the 0-brane is one dimensional and the solutions of the field equations namely (33) gives the geometric structure of the wordline originated from the currents on the brane defined from the symmetries of the global spacetime. All KY 1-forms define conserved currents on 0-branes and we have six different possibilities to construct a current. Different currents induce different localized curvatures around the branes.

For the second current $\mathcal{J}_2$, the field equations are changed only by a constant factor; because, the main difference coming from $\mathcal{J}_2$ is the addition of a scalar curvature term which is constant for AdS spacetime as can be seen from (17).

In fact, there is one more possibility to construct a conserved current using two different (or same) KY forms. From the conservation properties of $*\mathcal{J}_1$ and $*\mathcal{J}_2$, it can be seen that the following $(2n - (p+q))$-form in n dimensions is also a conserved current;

$$\mathcal{K}_{(2n-(p+q))} = *\mathcal{J}_i(\omega_{(p)}) \wedge *\mathcal{J}_j(\omega'_{(q)}) \quad (34)$$

where $\omega_{(p)}$ and $\omega'_{(q)}$ are two different (or same) KY forms and $i, j = 1, 2$. In three dimensions, this current is written as follows;

$$\mathcal{K}_{(2)} = *\mathcal{J}_1(\omega_{(2)}) \wedge *\mathcal{J}_1(\omega'_{(2)}). \quad (35)$$

Hence, in the construction procedure of gravitational conserved currents in three dimensions, KY 2-forms can

also be used besides KY 1-forms. By taking two KY 2-forms as

$$\begin{aligned}\omega_{(2)} &= \rho e^0_{AdS}\wedge e^1_{AdS} + \epsilon e^0_{AdS}\wedge e^2_{AdS} + \mu e^1_{AdS}\wedge e^2_{AdS}\\ \omega'_{(2)} &= \nu e^0_{AdS}\wedge e^1_{AdS} + \kappa e^0_{AdS}\wedge e^2_{AdS} + \lambda e^1_{AdS}\wedge e^2_{AdS}\end{aligned}$$

the field equations resulting from the current (35) are obtained as below;

$$\begin{aligned}R^{01} + \frac{1}{l^2}e^0\wedge e^1 &= \frac{8}{l^4}\left(\mu\kappa - \epsilon\lambda\right)e^0_{AdS}\wedge e^1_{AdS}\\ R^{02} + \frac{1}{l^2}e^0\wedge e^2 &= \frac{8}{l^4}\left(\rho\lambda - \mu\nu\right)e^0_{AdS}\wedge e^2_{AdS}\\ R^{12} + \frac{1}{l^2}e^1\wedge e^2 &= \frac{8}{l^4}\left(\rho\kappa - \epsilon\nu\right)e^1_{AdS}\wedge e^2_{AdS}\end{aligned}\quad (36)$$

where $\rho, \epsilon, \mu, \nu, \kappa$ and λ are functions obtained from the KY equation as in Appendix A.

B. Brane Couplings in 5D

In five dimensions, there are two possibilities of coupling a conserved current with a CS form. 4-form currents and 2-form currents can couple consistently with CS gravity.

For the coupling of a 4-form current on the brane with CS gravity, the action is written as

$$I_5 = \int_{M^5}\langle \mathcal{C}_5(\mathbf{A}) - j_{(4)}\wedge\mathbf{A}\rangle \quad (37)$$

and the field equations become

$$\mathbf{F}\wedge\mathbf{F} = j_{(4)}. \quad (38)$$

In the regions exterior to the brane, the equations can be solved by $\mathbf{F} = 0$ which gives the AdS spacetime. However, other solutions that satisfy $\mathbf{F}\wedge\mathbf{F} = 0$ can also appear.

The first KY current leads to a 4-form current $*\mathcal{J}_1$ in terms of KY 1-forms and the field equations take the form

$$\begin{aligned}&\frac{1}{4}\left(R^{ab} + \frac{1}{l^2}e^a\wedge e^b\right)\wedge\left(R^{cd} + \frac{1}{l^2}e^c\wedge e^d\right)[\mathbf{J}_{ab},\mathbf{J}_{cd}]\\ &= \left[*\left(-i_{X_k}\omega_{(1)}\wedge P^k_G\right)\right]^{abcd}\mathbf{J}_{ab}\mathbf{J}_{cd}\end{aligned}\quad (39)$$

and for the KY current $\mathcal{J}_2$ the field equations are

$$\begin{aligned}&\frac{1}{4}\left(R^{ab} + \frac{1}{l^2}e^a\wedge e^b\right)\wedge\left(R^{cd} + \frac{1}{l^2}e^c\wedge e^d\right)[\mathbf{J}_{ab},\mathbf{J}_{cd}]\\ &= \left[*\left(-i_{X_k}\omega_{(1)}\wedge P^k_G + \mathcal{R}_G\omega_{(1)}\right)\right]^{abcd}\mathbf{J}_{ab}\mathbf{J}_{cd}.\end{aligned}\quad (40)$$

In the case of 2-form currents coupled with CS gravity, the action will be

$$I_5 = \int_{M^5}\langle \mathcal{C}_5(\mathbf{A}) - j_{(2)}\wedge\mathcal{C}_3(\mathbf{A})\rangle \quad (41)$$

and the field equations are

$$\mathbf{F}\wedge\mathbf{F} = j_{(2)}\wedge\mathbf{F}. \quad (42)$$

Hence the first KY current $\mathcal{J}_1$ leads to the field equations

$$\begin{aligned}&\frac{1}{4}\left(R^{ab} + \frac{1}{l^2}e^a\wedge e^b\right)\wedge\left(R^{cd} + \frac{1}{l^2}e^c\wedge e^d\right)[\mathbf{J}_{ab},\mathbf{J}_{cd}]\\ &= \left[*\left(-i_{X_k}i_{X_l}\omega_{(3)}\wedge R^{kl}_G - i_{X_k}\omega_{(3)}\wedge P^k_G\right)\right]^{ab}\\ &\wedge\frac{1}{2}\left(R^{cd} + \frac{1}{l^2}e^c\wedge e^d\right)[\mathbf{J}_{ab},\mathbf{J}_{cd}]\end{aligned}\quad (43)$$

and the equations obtained from the second current $\mathcal{J}_2$ are

$$\begin{aligned}&\frac{1}{4}\left(R^{ab} + \frac{1}{l^2}e^a\wedge e^b\right)\wedge\left(R^{cd} + \frac{1}{l^2}e^c\wedge e^d\right)[\mathbf{J}_{ab},\mathbf{J}_{cd}]\\ &= \left[*\left(-i_{X_k}\omega_{(3)}\wedge P^k_G + \mathcal{R}_G\omega_{(3)}\right)\right]^{ab}\\ &\wedge\frac{1}{2}\left(R^{cd} + \frac{1}{l^2}e^c\wedge e^d\right)[\mathbf{J}_{ab},\mathbf{J}_{cd}].\end{aligned}\quad (44)$$

KY 2-forms and 4-forms in five dimensions can also be used instead of KY 1-forms and 3-forms in the construction of conserved currents, so from (34) we obtain;

$$\begin{aligned}\mathcal{K}_{(4)} &= *\mathcal{J}_i(\omega_{(2)})\wedge *\mathcal{J}_j(\omega_{(4)})\\ \mathcal{K}_{(2)} &= *\mathcal{J}_i(\omega_{(4)})\wedge *\mathcal{J}_j(\omega'_{(4)}).\end{aligned}$$

In five dimensions, 0-branes and 2-branes can couple consistently with CS theories as can be seen respectively from the actions (37) and (41). Solutions of the field equations give the local geometries on the worldvolumes of the branes.

VI. CONCLUSION

Generalization of minimal coupling procedure for external sources to p-brane spacetimes can not be done by extending the coupling term to multi-index currents and connections in non-abelian gauge theories. But, the coupling can be considered consistently if one uses CS forms in the coupling term. This can be relevant because of the abelian gauge transformation property of the CS forms. CS theories are defined in odd dimensions and because of the metric independence of the action, they are topological theories. By selecting the gauge connection as an AdS connection, which includes coframe and spin connection, CS theories of gravity can be constructed in odd dimensions. Hence, coupling of electromagnetic conserved currents on p-branes and CS gravities can be consistently considered in this fashion.

For curved backgrounds, one can construct gravitational conserved currents by using curvature characteristics and KY forms of spacetime. These currents depend on the degree of the KY form and this allows to the interpretation that they are localized on p-branes. Hence,

the coupling of gravitational p-brane currents with CS gravities can be considered in the same manner as in the electromagnetic case. The field equations resulting from the coupling actions give that the one solution is an AdS spacetime for the spacetime regions exterior to the brane. This means that the gravitational currents are constructed from the AdS curvature and KY forms. So, the field equations give the local curvature on p-branes induced by gravitational currents.

In three dimensional case, the field equations tells us that the localized curvature on branes has correction terms with respect to the AdS background written in terms of KY form components. In five dimensional case, there are two different couplings and they end up with different field equations for different branes. However, resulting equations also give the localized curvature on the branes.

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Appendix A: KY Forms of AdS Spacetime in 3D

KY forms of a class of spherically symmetric spacetimes in four dimensions are found in [10] by solving the KY equation. By direct reduction, KY forms of three dimensional spacetimes can also be obtained from them. Metric tensor field of AdS spacetime in three dimensions is

$$ds^2_{AdS} = -\left(\frac{r^2}{l^2}+1\right)dt^2 + \left(\frac{r^2}{l^2}+1\right)^{-1}dr^2 + r^2 d\phi^2 \tag{A1}$$

and this can be written in a locally Lorentzian form as follows;

$$ds^2_{AdS} = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 \tag{A2}$$

where

$$e^0 = H_0 dt \quad , \quad e^1 = H_1 dr \quad , \quad e^2 = r d\phi \tag{A3}$$

and

$$H_0 = \left(\frac{r^2}{l^2}+1\right)^{1/2} \quad , \quad H_1 = \left(\frac{r^2}{l^2}+1\right)^{-1/2} . \tag{A4}$$

In three dimensions the maximal number of KY 1-forms is six, which can be seen from (18). The corresponding KY 1-forms of AdS spacetime are

$$\omega_1 = H_0 e^0 \tag{A5}$$

$$\omega_2 = -re^2 \tag{A6}$$

$$\omega_3 = (\cos\phi)\psi_1 - \frac{\kappa}{H_1}\sinh(\kappa t)\sin\phi e^2 \tag{A7}$$

$$\omega_4 = (\cos\phi)\psi_2 - \frac{\kappa}{H_1}\cosh(\kappa t)\sin\phi e^2 \tag{A8}$$

$$\omega_5 = -(\sin\phi)\psi_1 - \frac{\kappa}{H_1}\sinh(\kappa t)\cos\phi e^2 \tag{A9}$$

$$\omega_6 = -(\sin\phi)\psi_2 - \frac{\kappa}{H_1}\cosh(\kappa t)\cos\phi e^2 \tag{A10}$$

where κ is an integration constant and ψ_1 and ψ_2 are defined as follows;

$$\psi_1 = \cosh(\kappa t)\frac{H_0'}{H_1}e^0 + \kappa\sinh(\kappa t)e^1 \tag{A11}$$

$$\psi_2 = \sinh(\kappa t)\frac{H_0'}{H_1}e^0 + \kappa\cosh(\kappa t)e^1 \tag{A12}$$

and $H_0' = \frac{dH_0}{dr}$.

The number of KY 2-forms is four and they are obtained as

$$\lambda_1 = \cos\phi e^0 \wedge e^1 - \frac{1}{H_1}\sin\phi e^0 \wedge e^2 \tag{A13}$$

$$\lambda_2 = \sin\phi e^0 \wedge e^1 + \frac{1}{H_1}\cos\phi e^0 \wedge e^2 \tag{A14}$$

$$\lambda_3 = -\frac{w_0}{m_1}\sinh(w_0 t)e^0 \wedge e^2 + \cosh(w_0 t)e^1 \wedge e^2 \tag{A15}$$

$$\lambda_4 = -\frac{w_0}{m_1}\cosh(w_0 t)e^0 \wedge e^2 + \sinh(w_0 t)e^1 \wedge e^2 \tag{A16}$$

where $m = H_0'/rH_1$, $m_1 = (r/H_0)'H_0^2/H_1$ and $mm_1 = \pm w_0^2$. In all dimensions, the volume form multiplied with a constant automatically satisfies the KY equation. Hence, the KY 3-form in three dimensions is

$$\omega_{(3)} = ce^0 \wedge e^1 \wedge e^2 \tag{A17}$$

where c is a constant.

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